Potential Flow

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Overview

- For irrotational flow, $\nabla \times \vec{V} = 0$, which implies that $\vec{V} = \pm \nabla \phi$.
- ϕ is a scalar field called the potential flow function.

- If the fluid is incompressible, then the continuity equation implies that $\nabla \cdot \vec{V} = 0$.
- In this case, the potential flow function satisfies the Laplace equation, $\nabla^2 \phi = 0$.

- We can obtain many velocity fields using the techniques used to solve Laplace's equation.

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Flow potential

Consider, $\mathrm{d}\phi = u_1 \mathrm{d}x_1 + u_2 \mathrm{d}x_2 + u_3 \mathrm{d}x_3.$

ϕ is a single valued function iff $\dfrac{\partial^2 \phi}{\partial x_1 \partial x_2} = \dfrac{\partial^2 \phi}{\partial x_2 \partial x_1},$ and two similar eqs. by exchanging 1 or 2 by 3.

which is equivalent to $\dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} = (\nabla \wedge \boldsymbol{u})_3 = \Omega_3 = 0,$ with similar eqs. for components 1 and 2.

meaning that, the flow is irrotational (i.e. the vorticity is zero).

For irrotational flows, the velocity field is the gradient of a scalar flow potential ϕ:

$$\boldsymbol{u}\{\boldsymbol{x}, t\} = \nabla \phi\{\boldsymbol{x}, t\},$$

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Velocity field

Given the flow potential, the velocity field is obtained by taking its gradient (recap):

Cartesian coordinates,

$$u = \frac{\partial \phi}{\partial x} \quad v = \frac{\partial \phi}{\partial y} \quad w = \frac{\partial \phi}{\partial z}$$

and in cylindrical coordinates,

$$u_r = \frac{\partial \phi}{\partial r} \quad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} \quad u_z = \frac{\partial \phi}{\partial z}$$

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Flow potential, incompressible flow

- From the continuity equation, we have

$$\rho \nabla \cdot \boldsymbol{u} = \rho \nabla^2 \phi = -\frac{\mathrm{D}\rho}{\mathrm{D}t}.$$

- If the flow is incompressible, then

$$\nabla^2 \phi = 0.$$

i.e., the flow potential is a solution of Laplace's equation.

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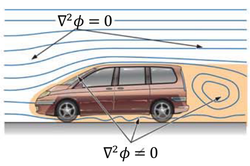
Example (schematic)



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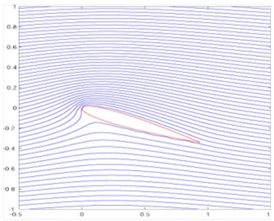
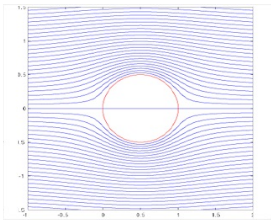
Examples (solutions of Laplace's equation)

Airfoil in a free stream



Cylinder in a free stream



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Examples from MFM (page 271)

Comparison of potential flow theory and experiments (hele-shaw cell) and experiments for flows with Re=10^4 .

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Back to Laplace's equation

For irrotational regions of flow: $\nabla^2\phi = 0$

In cartesian coordinates

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

In cylindrical coordinates

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Spherical and mixed coordinates may also be useful.

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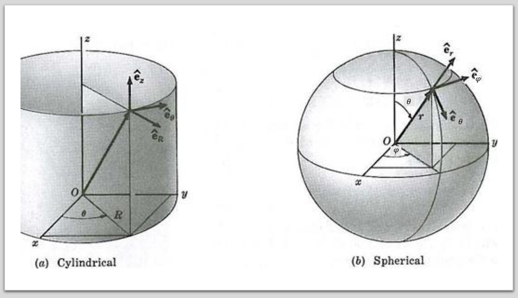

Cylindrical and spherical coordinates

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Cylindrical Coordinates (r, θ, z)

$$r^2 = x^2 + y^2,\ \theta = \tan^{-1}(y/x)$$

$$\vec{V} = u_r\hat{e}_r + u_\theta\hat{e}_\theta + u_z\hat{e}_z = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{\partial\phi}{\partial z}\hat{e}_z = \nabla\phi$$

$$\nabla^2\phi = \underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Cartesian Coordinates (x, y, z)

$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k} = \nabla\phi$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

Spherical Coordinates (r, θ, φ)

$$r^2 = x^2 + y^2 + z^2,\ \theta = \cos^{-1}(x/r),\ \text{or}\ x = r\cos\theta,\ \varphi = \tan^{-1}(z/y)$$

$$\vec{V} = u_r\hat{e}_r + u_\theta\hat{e}_\theta + u_\varphi\hat{e}_\varphi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\hat{e}_\varphi = \nabla\phi$$

$$\nabla^2\phi = \underbrace{\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r}}_{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\varphi^2} = 0$$

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- The beauty of this is that we have combined three unknown velocity components (e.g., u, v, and w) into one unknown scalar field ϕ, eliminating two of the equations required for a solution.
- Once we obtain a solution, we can calculate all three components of the velocity field.
- The Laplace equation is well known since it shows up in several fields of physics, applied mathematics, and engineering. Various solution techniques, both analytical and numerical, are available in the literature.
- Solutions of the Laplace equation are dominated by the geometry (i.e., boundary conditions).
- The solution is valid for any incompressible fluid, regardless of its density or its viscosity, in regions of the flow in which the irrotational approximation is appropriate

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Pressure

Of course we still need a dynamical equation to calculate the pressure field. This will be given by the Euler equation (this is the form of the Navier-Stokes equation for irrotational flow – see later).

If gravity is the only body force, then

For irrotational regions of flow: $$\rho\left[\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\vec{\nabla})\vec{V}\right]=-\vec{\nabla}P+\rho\vec{g}$$

Or in its integrated form, the Bernoulli equation

Steady, incompressible flow: $$\frac{P_1}{\rho}+\frac{V_1^2}{2}+gz_1=\frac{P_2}{\rho}+\frac{V_2^2}{2}+gz_2$$

Since the flow is irrotational, we can apply Bernoulli to ANY two points in the flow domain.

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Why are the solutions of Laplace´s equation useful (at all) ?

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Kelvin's circulation theorem

- A fluid that is vorticity free at a given instant is vorticity free at all times.
- Demonstration: see Faber 120-122 (you may skip this on a first reading)
- In three dimensions the conservation of vorticity (which corresponds to the conservation of angular momentum in mechanics) takes a somewhat subtle form.
- The circulation of a velocity field is defined to be

$$K\{t\} = \oint \boldsymbol{u}\{\boldsymbol{x},t\}\cdot \mathrm{d}\boldsymbol{l},$$

where the line is a closed loop which moves with the fluid.

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Circulation and vorticity

- By Stoke's theorem

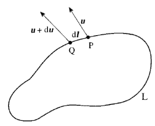
$$K=\oint_{C(t)}\mathbf{u}\cdot d\mathbf{l}=\int_{S(t)}(\boldsymbol{\nabla}\times\mathbf{u})\cdot\mathbf{n}dS=\int_{S(t)}\Omega\cdot\mathbf{n}dS,$$

where S(t) is a surface whose edges connect with C(t).

K is zero for all loops if Ω is zero in the domain!

Kelvin´s theorem asserts that

$$\frac{DK}{Dt}=0.$$



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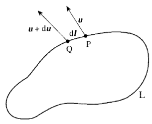
Demonstration

The loop moves with the flow and thus



$$\frac{\mathrm{D}K}{\mathrm{D}t}=\oint\left\{\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot\mathrm{d}\boldsymbol{l}+\boldsymbol{u}\cdot\frac{\mathrm{D}(\mathrm{d}\boldsymbol{l})}{\mathrm{D}t}\right\}\cdot$$

The second term is the relative velocity of two nearby points on the loop and can be written as $(\partial\boldsymbol{u}/\partial l)\mathrm{d}l$.

$$\boldsymbol{u}\cdot\frac{\mathrm{D}(\mathrm{d}\boldsymbol{l})}{\mathrm{D}t}=\boldsymbol{u}\cdot\frac{\partial\boldsymbol{u}}{\partial l}\,\mathrm{d}l=\frac{\partial(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u})}{\partial l}\,\mathrm{d}l=\mathrm{d}\left(\frac{1}{2}u^2\right)\cdot$$

The second term integrates to zero and the first can be re-written using Euler

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=-\left\{\frac{1}{\rho}\boldsymbol{\nabla}p+\boldsymbol{\nabla}(gz)\right\}\cdot$$

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If the fluid is incompressible,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}=-\boldsymbol{\nabla}\left(\frac{p}{\rho}+gz\right),$$

and the first term in the integral

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t}\cdot\mathrm{d}\boldsymbol{l}=-\mathrm{d}\left(\frac{p}{\rho}+gz\right)\cdot$$

also integrates to zero, proving the result

$$\frac{\mathrm{D}K}{\mathrm{D}t}=0$$

This means that if K is zero at some time it will remain so for all t.

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Back to the solutions of Laplace's equation

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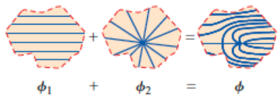
Superposition

- Since the Laplace equation is a linear homogeneous differential equation, the linear combination of two or more solutions of the equation must also be a solution.
- For example, if ϕ_1 and ϕ_2 are each solutions of the Laplace equation, then $A\,\phi_1 + B\,\phi_2$ are also solutions, where A and B are arbitrary constants.
- By extension, you may combine several solutions of the Laplace equation, and the combination is guaranteed to also be a solution.



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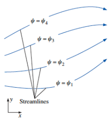
Planar flows: Stream function

- For planar incompressible flows we can also define the stream function, ψ.

Incompressible, two-dimensional stream function in Cartesian coordinates:

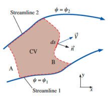
$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

- Curves of constant ψ are streamlines of the flow (see later)



The difference in the value of ψ from one streamline to another is equal to the volume flow rate per unit width between the two streamlines.

$$\dot{V}_B = \int_B \vec{V}\cdot\vec{n}\,dA = \int_B d\dot{V} = \int_{\psi=\psi_1}^{\psi=\psi_2} d\psi = \psi_2 - \psi_1$$



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Planar and axisymmetric flows

The stream function is defined for incompressible (divergence-free) flows in two dimensions – as well as in three dimensions with axisymmetry (2 independent variables).

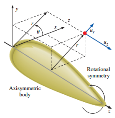
The flow velocity components can also be expressed as derivatives of the scalar stream function.

Incompressible, planar stream function in cylindrical coordinates:

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

Incompressible, axisymmetric stream function in cylindrical coordinates:

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial z} \quad \text{and} \quad u_z = \frac{1}{r}\frac{\partial \psi}{\partial r}$$



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Equation for the stream function

- For irrotational flows in 2D, the stream function obeys the Laplace equation:

$$\nabla^2 \psi = 0.$$

- Thus in potential 2D flow, both the flow potential and the stream function are solutions of the Laplace equation.
- Lines of constant flow potential are perpendicular to the streamlines (easily checked).
- In axisymmetric flows the stream function obeys a linear equation but that is no longer Laplace's equation.

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Uniform (free) stream

Uniform stream: $\quad u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = V \quad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = 0$

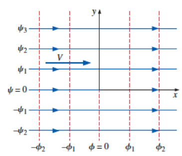
$$\phi = Vx + f(y) \quad \rightarrow \quad v = \frac{\partial \phi}{\partial y} = f'(y) = 0 \quad \rightarrow \quad f(y) = \text{constant}$$

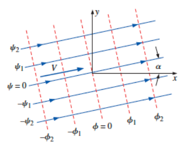
Velocity potential function for a uniform stream: $\quad \phi = Vx$

Stream function for a uniform stream: $\quad \psi = Vy$

Uniform stream: $\quad \phi = Vr\cos\theta \quad \psi = Vr\sin\theta$

Uniform stream inclined at angle α: $\quad \phi = V(x\cos\alpha + y\sin\alpha)$, $\quad \psi = V(y\cos\alpha - x\sin\alpha)$





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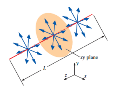
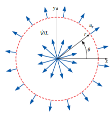
Line source or sink

Let the volume flow rate per unit depth, be the line source strength, m



$$\frac{\dot{V}}{L} = 2\pi r u_r, \quad u_r = \frac{\dot{V}/L}{2\pi r}$$

The components of the velocity are

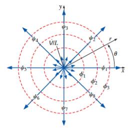


Line source: $u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\dot{V}/L}{2\pi r} \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = 0$

$$\frac{\partial \psi}{\partial r} = -u_\theta = 0 \quad \rightarrow \quad \psi = f(\theta) \quad \rightarrow \quad \frac{\partial \psi}{\partial \theta} = f'(\theta) = r u_r = \frac{\dot{V}/L}{2\pi}$$

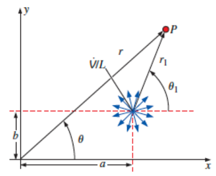
With solution

$$f(\theta) = \frac{\dot{V}/L}{2\pi}\theta + \text{constant}$$



Line source at the origin: $\phi = \frac{\dot{V}/L}{2\pi}\ln r$ and $\psi = \frac{\dot{V}/L}{2\pi}\theta$

Line source or sink at an arbitrary point



Line source at point (a, b):

$$\phi = \frac{\dot{V}/L}{2\pi}\ln r_1 = \frac{\dot{V}/L}{2\pi}\ln\sqrt{(x-a)^2 + (y-b)^2}$$

$$\psi = \frac{\dot{V}/L}{2\pi}\theta_1 = \frac{\dot{V}/L}{2\pi}\arctan\frac{y-b}{x-a}$$

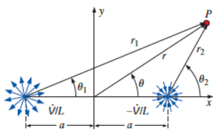
Superposition of a source and sink of equal strength

Line source at $(-a, 0)$: $\psi_1 = \frac{\dot{V}/L}{2\pi}\theta_1$ where $\theta_1 = \arctan\frac{y}{x+a}$

Similarly for the sink,

Line sink at $(a, 0)$: $\psi_2 = \frac{-\dot{V}/L}{2\pi}\theta_2$ where $\theta_2 = \arctan\frac{y}{x-a}$

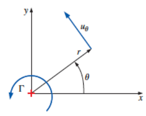
Composite stream function: $\psi = \psi_1 + \psi_2 = \frac{\dot{V}/L}{2\pi}(\theta_1 - \theta_2)$



Final result, Cartesian coordinates: $\psi = \frac{-\dot{V}/L}{2\pi}\arctan\frac{2ay}{x^2 + y^2 - a^2}$

Final result, cylindrical coordinates: $\psi = \frac{-\dot{V}/L}{2\pi}\arctan\frac{2ar\sin\theta}{r^2 - a^2}$

Line vortex



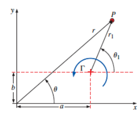
The radial component of the velocity is zero and

Line vortex: $u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0 \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$

where $\Gamma = 2\pi r u_\theta$, is the circulation, around a loop of radius r.

Then,

Line vortex at the origin: $\phi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r$



Line vortex at point (a, b):

$$\phi = \frac{\Gamma}{2\pi}\theta_1 = \frac{\Gamma}{2\pi}\arctan\frac{y-b}{x-a}$$

$$\psi = -\frac{\Gamma}{2\pi}\ln r_1 = -\frac{\Gamma}{2\pi}\ln\sqrt{(x-a)^2 + (y-b)^2}$$

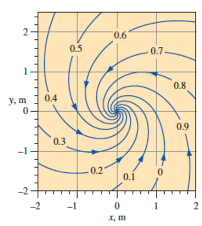
Superposition of a line sink and a line vortex at the origin

The stream function is

Superposition: $\psi = \frac{\dot{V}/L}{2\pi}\theta - \frac{\Gamma}{2\pi}\ln r$

with streamlines

Streamlines: $r = \exp\left(\frac{(\dot{V}/L)\theta - 2\pi\psi}{\Gamma}\right)$



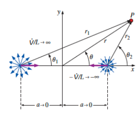
Velocity components: $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\dot{V}/L}{2\pi r} \qquad u_\theta = -\frac{\partial \psi}{\partial r} = \frac{\Gamma}{2\pi r}$

Note that velocity diverges at the origin, which is a singularity (unphysical).

Doublet: line source and sink close to origin

We have seen before (slide 23) that

Composite stream function: $\psi = \frac{-\dot{V}/L}{2\pi}\arctan\frac{2ar\sin\theta}{r^2 - a^2}$



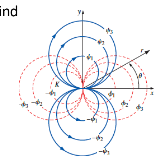
By Taylor expanding the arctan around zero:

Stream function as $a \to 0$: $\psi \to \frac{-a(\dot{V}/L)r\sin\theta}{\pi(r^2 - a^2)}$

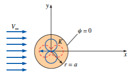
Let a tend to zero at constant doublet strength K, to find

Doublet along the x-axis: $\psi = \frac{-a(\dot{V}/L)}{\pi}\frac{\sin\theta}{r} = -K\frac{\sin\theta}{r}$

Doublet along the x-axis: $\phi = K\frac{\cos\theta}{r}$



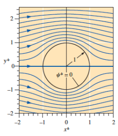
Superposition of a uniform stream and a doublet: Flow over a circular cylinder



Superposition: $\psi = V_\infty r \sin\theta - K\frac{\sin\theta}{r}$

For convenience we set $\psi = 0$ when $r = a$

Doublet strength: $K = V_\infty a^2$

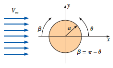


Alternate form of stream function: $\psi = V_\infty \sin\theta\left(r - \frac{a^2}{r}\right)$

$$\psi^* = \sin\theta\left(r^* - \frac{1}{r^*}\right)$$

Nondimensional streamlines: $r^* = \frac{\psi^* \pm \sqrt{(\psi^*)^2 + 4\sin^2\theta}}{2\sin\theta}$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = V_\infty\cos\theta\left(1 - \frac{a^2}{r^2}\right) \quad u_\theta = -\frac{\partial\psi}{\partial r} = -V_\infty\sin\theta\left(1 + \frac{a^2}{r^2}\right)$$



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3D Flow

- The 1/R potential $\phi = -\frac{Q}{4\pi R}$ is a solution of Laplace´s equation in 3D
- It describes isotropic flow with velocity $Q/4\pi R^2$
- If Q > 0 it is a source and it is a sink otherwise. Q is the discharge rate.
- Free stream potential $\phi = Ux_1.$
- Superposition of the two gives

$$u_1 = U + \frac{Q}{4\pi R^2}\cos\theta, \quad (u_2^2 + u_3^2)^{1/2} = \frac{Q}{4\pi R^2}\sin\theta,$$

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- Or in spherical coordinates,

$$u_R = U\cos\theta + \frac{Q}{4\pi R^2}, \quad u_\theta = -U\sin\theta.$$

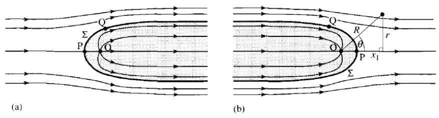

Figure 4.2 Lines of flow past (a) a point source, (b) a point sink. The surface of revolution Σ encloses all the fluid coming from, or destined for, the source or sink respectively.

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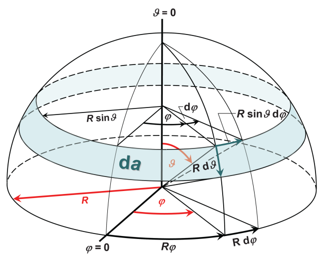
Reminder

Spherical coordinates



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Excess pressure and force

The excess pressure vanishes at infinity where the velocity is that of the free stream. Then Bernoulli gives for the dynamical pressure:

$$p^* = \frac{1}{2}\rho(U^2 - u_R^2 - u_\theta^2) = -\frac{\rho UQ\cos\theta}{4\pi R^2} - \frac{\rho Q^2}{32\pi^2 R^4}.$$

The total force in the 1 (stream) direction exerted by the excess pressure on the fluid inside a spherical control surface, centred at O, of arbitrary radius R, is

$$-\int_0^\pi p^* 2\pi R^2 \sin\theta \, d\theta \cos\theta,$$

where $2\pi R^2 \sin\theta \, d\theta$ is the area of a ring shaped element on the surface of the sphere and cosθ gives the projection of the force in the 1 direction.

The total force is, after integration,

$$\frac{1}{2}\rho UQ\int_0^\pi\left(\cos^2\theta\sin\theta + \frac{Q\cos\theta\sin\theta}{8\pi R^2 U}\right)d\theta = \frac{1}{3}\rho UQ.$$

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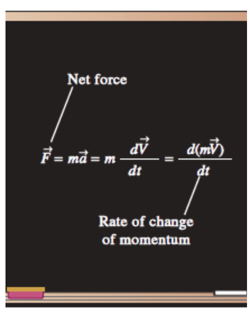

Newton´s second law for a control volume

Fixed CV: $\sum\vec{F} = \frac{d}{dt}\int_{CV}\rho\vec{V}\,d\mathcal{V} + \int_{CS}\rho\vec{V}(\vec{V}\cdot\vec{n})\,dA$

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Rate of change of momentum

- The total force is equal to the rate of change of momentum in the 1 direction of the fluid, within the sphere:

$$\int_0^{\pi} \rho u_1 u_R 2\pi R^2 \sin\theta \, d\theta$$
$$= \int_0^{\pi} \left\{ U^2 \cos\theta + \frac{UQ(1+\cos^2\theta)}{4\pi R^2} + \frac{Q^2 \cos\theta}{16\pi^2 R^4} \right\} 2\pi R^2 \sin\theta \, d\theta$$
$$= \frac{4}{3} \rho U Q ,$$

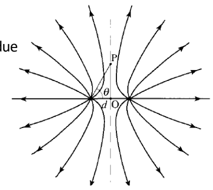
- There is then an additional force on the fluid in the 1 direction of magnitude $\rho U Q$
- This has to be exerted by the source (sink) and thus the source (sink) will experience a reaction force

$$\boldsymbol{F} = -\rho U Q .$$

Two equal sources

Velocity at one source, due to the other:

$U = Q/4\pi(2d)^2$



$$OP = r = \frac{d}{\cos\theta} = d \sec\theta$$

On the plane bissecting the line joining the two sources the normal component of the velocity vanishes. The radial component (in the direction of OP), add and are given by:

$$\frac{2Q \sin\theta}{4\pi(d \sec\theta)^2} .$$

Excess pressure and force

- Assuming that the excess pressure vanishes at infinity, where **u** also vanishes, the excess pressure at P is (Bernoulli),

$$p^*\{\theta\} = -\frac{\rho Q^2 \sin^2\theta \cos^4\theta}{8\pi^2 d^4} .$$

- The fluid to the left of the bissecting plane experiences a force (to the right) due to this excess pressure, given by

$$-\int_0^{\infty} p^*\{\theta\} 2\pi d \tan\theta \, d(d \tan\theta) = \frac{\rho Q^2}{4\pi d^2} \int_0^{\pi/2} \sin^3\theta \cos\theta \, d\theta = \frac{\rho Q^2}{16\pi d^2} .$$

Since the effect of the flow is to transfer fluid to infinity from whatever reservoirs supply the sources, and since the fluid at infinity, like that in the reservoirs, has zero momentum, the total momentum of the fluid on either side of the bisecting plane does not change with time. It follows that the whole of the force calculated above must be transferred by the fluid to the source enclosed within it. Hence the source on the left is drawn to the right (and *vice versa*), and the strength of the attraction is

$$\frac{\rho Q^2}{16\pi d^2} = \rho U Q .$$

Analytical solutions of Laplace's equation

(i) *Two-dimensional circular polar coordinates (r, θ)*

In this system Laplace's equation becomes

$$r \frac{\partial}{\partial r}\left\{ r \frac{\partial \phi}{\partial r} \right\} + \frac{\partial^2 \phi}{\partial \theta^2} = 0.$$

Single-valued solutions in which the variables are separated can readily be found. They are:

$$\phi = \text{constant},$$
$$\phi \propto \phi_0 = \ln r, \qquad (4.22)$$
$$\phi \propto \phi_n = r^n \cos(n\theta), \quad or \quad \phi \propto \psi_n = r^n \sin(n\theta) \qquad (4.23)$$
$$[n = \pm 1, \pm 2, \pm 3 \text{ etc.}].$$

$$\phi = \text{constant} + A_0 \phi_0 + \Sigma(A_n \phi_n + B_n \psi_n)$$

(ii) *Complex potentials in two dimensions*

A powerful general method for handling Laplace's equation in two dimensions rests on some elementary results in the theory of functions of complex variables. Let

$$\phi + i\psi = f\{x_1 + i x_2\}, \qquad (4.25)$$

where ϕ and ψ are both real and where $f\{Z\}$ is any sensible, differentiable, function of its argument Z. Then

$$\frac{\partial^2 \phi}{\partial x_1^2} = f'', \quad \frac{\partial^2 \phi}{\partial x_2^2} = i^2 f'' = -f'',$$

so

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = 0,$$

i.e. ϕ is a solution of Laplace's equation in the two-dimensional space covered by the cartesian coordinates (x_1, x_2), and the same is true of ψ. Now

$$(\boldsymbol{\nabla}\phi)\cdot(\boldsymbol{\nabla}\psi) = \frac{\partial \phi}{\partial x_1}\frac{\partial \psi}{\partial x_1} + \frac{\partial \phi}{\partial x_2}\frac{\partial \psi}{\partial x_2}$$
$$= i^{-1}(f')^2 + i(f')^2$$
$$= 0.$$

Hence the two-dimensional gradient vectors $\nabla\phi$ and $\nabla\psi$ are everywhere orthogonal to one another, and, since they are orthogonal to the contours of constant ϕ and ψ respectively, these contours must be orthogonal to one another also. If we choose to regard the quantity ϕ defined by (4.25) as a two-dimensional flow potential such that $\nabla\phi = \boldsymbol{u}$, then $\boldsymbol{u}$ is tangential everywhere to the contours of constant ψ, and these contours therefore serve to describe the streamlines associated with ϕ or, in cases of steady flow, the lines of flow.

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(iv) *Three-dimensional spherical polar coordinates* (R, θ, ϕ)

Laplace's equation in spherical polars has separated solutions which form a complete set, like the two-dimensional solutions described by (4.22) and (4.23). We need not list them fully here, because we shall be concerned only with problems in which the flow is axially symmetric, i.e. in which the flow potential does not vary with the azimuthal angle ϕ.[2] In these circumstances Laplace's equation simplifies to

$$\frac{\partial}{\partial R}\left(R^2 \frac{\partial\phi}{\partial R}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\phi}{\partial\theta}\right) = 0,$$

and its separated solutions may be written as

$$\phi \propto \phi_n^+ = R^n\,\mathrm{P}_n\{\cos\theta\}, \tag{4.27}$$

$$\phi \propto \phi_n^- = R^{-(n+1)}\,\mathrm{P}_n\{\cos\theta\}, \tag{4.28}$$

$$[n = 0, +1, +2, +3 \text{ etc.}].$$

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The Legendre functions $\mathrm{P}_n\{\cos\theta\}$ may be expanded as polynomials in their argument, and we shall need the following expressions in particular:

$$\mathrm{P}_0\{\cos\theta\} = 1, \tag{4.29}$$

$$\mathrm{P}_1\{\cos\theta\} = \cos\theta, \tag{4.30}$$

$$\mathrm{P}_2\{\cos\theta\} = \frac{1}{2}(3\cos^2\theta - 1). \tag{4.31}$$

The full functions ϕ_n^+ and ϕ_n^- are properly called *zonal solid harmonics*. They are orthogonal to one another, and all other solutions of Laplace's equation in three dimensions which share their symmetry (or asymmetry) may be expressed as linear combinations of them [cf. (4.24)].

Some of the solutions described by (4.27) and (4.28) are of course trivial. Thus $\phi_0^+ = 1$ for all values of R and θ. As for

$$\phi_1^+ = R\cos\theta = x_1$$

and

$$\phi_0^- = R^{-1},$$

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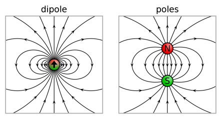
Dipolar flow

The function ϕ_0^- multiplied by $\mu_0 m/4\pi$ describes the field around a monopole of strength m at the origin.



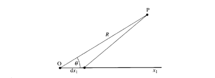


Figure 4.5 Displacement of O through Δx_1 changes the length of OP by $\Delta R = -\cos\theta\,\Delta x_1$ and changes its inclination by $\Delta\theta = +\sin\theta\,\Delta x_1/R$.

A magnetic dipole of strength $\mathrm{M} = m\Delta x_1$ produces the dipolar field:

$$\frac{\mu_0 m}{4\pi}\left\{-\frac{\partial\phi_0^-\{\boldsymbol{x}\}}{\partial x_1}\left(\frac{1}{2}\Delta x_1\right) + \frac{\partial\phi_0^-\{\boldsymbol{x}\}}{\partial x_1}\left(-\frac{1}{2}\Delta x_1\right)\right\} = -\frac{\mu_0 M}{4\pi}\frac{\partial\phi_0^-\{\boldsymbol{x}\}}{\partial x_1},$$

Or in spherical polar coordinates (see figure):

$$-\frac{\mu_0 M}{4\pi}\left(\cos\theta\,\frac{\partial\phi_0^-}{\partial R} - \frac{1}{R}\sin\theta\,\frac{\partial\phi_0^-}{\partial\theta}\right) = \frac{\mu_0 m\Delta x_1}{4\pi}\frac{1}{R^2}\cos\theta = \frac{\mu_0 M}{4\pi}\phi_1^-.$$

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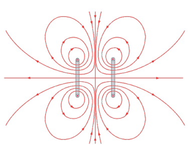
Higher multipoles

Thus in so far as ϕ_0^- may be described as *monopolar* ϕ_1^- is *dipolar*, and since ϕ_2^- may be related in a similar fashion to $\partial\phi_1^-/\partial x_1$ – there exists a useful recurrence relation

$$\phi_{n+1}^- = -\frac{1}{n+1}\frac{\partial\phi_n^-}{\partial x_1} = -\frac{1}{n+1}\left(\cos\theta\,\frac{\partial\phi_n^-}{\partial R} - \frac{1}{R}\sin\theta\,\frac{\partial\phi_n^-}{\partial\theta}\right) \tag{4.32}$$

which may readily be shown to hold for all values of n – ϕ_2^- is *quadrupolar*. This terminology derives from magnetostatics and electrostatics, but it is used in fluid dynamics too.



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Potential flow around a sphere

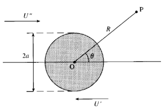

Figure 4.6 Coordinate system for discusssion of flow past a sphere. The sphere moves to the left with velocity U' and the fluid at large distances from the sphere moves to the right with uniform velocity U''. Their relative velocity is $U' + U'' = U$.

Expansion of the solution in zonal spherical harmonics

$$\phi = \Sigma(A_n^+\phi_n^+ + A_n^-\phi_n^-).$$

Boundary condition that at infinity in the $\theta = 0$ the velocity is that of the free stream implies that $A_1^+ = U''$ and $A_n^+ = 0$ for all $n > 1$.

At the surface of the sphere *R = a,* contact between the sphere and fluid require that the radial component of the fluid velocity is the same as that of the sphere

$$\left(\frac{\partial\phi}{\partial R}\right)_{R=a} = -U'\cos\theta.$$

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The latter implies that the potential has no monopolar or higher order multipolar terms,

$$\phi = A_0^+ + U''R\cos\theta + A_2^- R^{-2}\cos\theta,$$

We can set the first term to zero without loss of generality and then

$$\phi = U''R\cos\theta + \frac{1}{2}a^3(U' + U'')\frac{\cos\theta}{R^2}\cdot$$

The corresponding velocity components of the fluid are

$$u_R = \frac{\partial\phi}{\partial R} = U''\cos\theta - U\left(\frac{a}{R}\right)^3\cos\theta$$

$$u_\theta = \frac{1}{R}\frac{\partial\phi}{\partial\theta} = -U''\sin\theta - \frac{1}{2}U\left(\frac{a}{R}\right)^3\sin\theta,$$

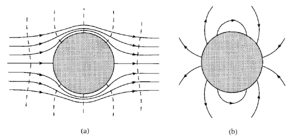

Figure 4.7 (a) Lines of flow and equipotentials round a stationary sphere ($U' = 0$). (b) The same flow pattern in a frame of reference such that $U'' = 0$ and the sphere is moving.

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Excess pressure

With p^*defined to be zero at large distances, we have

$$p^* = \frac{1}{2}\rho(U^2 - u_R^2 - u_\theta^2),$$

so that in contact with the sphere

$$p^*_{R=a} = \frac{1}{2}\rho U^2\left(1 - \frac{9}{4}\sin^2\theta\right)\cdot$$

Because the excess pressure at $R = a$ is completely symmetrical about the equatorial plane, a sphere which is in uniform motion relative to fluid experiences no force, apart from its own weight and the hydrostatic upthrust which we have suppressed. This is an example of *d'Alembert's paradox*

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Lift & drag forces

- The component of the resultant pressure and shear forces that acts in the flow direction is called the drag force (or just drag), and the component that acts normal to the flow direction is called the lift force (or just lift).

Drag force: $F_D = \int_A dF_D = \int_A (-P\cos\theta + \tau_w \sin\theta)\, dA$

and

Lift force: $F_L = \int_A dF_L = -\int_A (P\sin\theta + \tau_w \cos\theta)\, dA$

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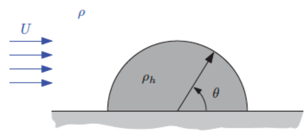

Lift on a half-sphere

- Due to high speed flow at the top of the sphere, we expect a low pressure at the top of the sphere. This pressure results in a lift force on the hemsiphere.

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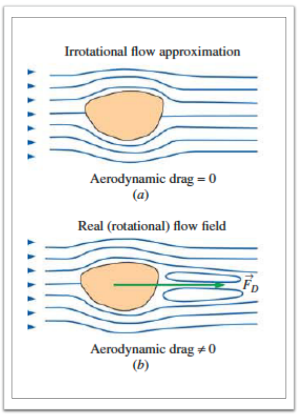

D'Alembert's paradox: In irrotational flow , the aerodynamic drag force on any body of any shape immersed in a uniform stream is zero.

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The Magnus effect (rotating cylinder)

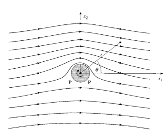
Figure 4.13 Potential flow past a transverse cylinder of radius a when the circulation round the cylinder is $K_3 = -2\pi Ua$. These lines of flow are contours of constant ψ, and the value of ψ increases from one line to the next by 1.

Superposition of solutions of Laplace's equation gives for the flow potential

$$\phi = U_1\cos\theta\left(r + \frac{a^2}{r}\right) + \frac{K_3\theta}{2\pi},$$

and the velocity

$$u_r = U_1\cos\theta\left(1 - \frac{a^2}{r^2}\right),\quad u_\theta = -U_1\sin\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{K_3}{2\pi r}\cdot$$

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The flow lines can be computed from the stream function,

$$\psi = U_1 \sin\theta \left(r - \frac{a^2}{r}\right) + \frac{K_3}{2\pi} \ln\left(\frac{r}{a}\right)$$

Bernoulli's theorem gives for the excess pressure:

$$p^*_{r=a} = \frac{1}{2}\rho(U_1^2 - u_\theta^2) = \frac{1}{2}\rho\left\{U_1^2 - \left(-2U_1\sin\theta + \frac{K_3}{2\pi a}\right)\right. = \frac{1}{2}\rho\left(\frac{K_3}{2\pi a}\right)^2 + \frac{1}{2}\rho U_1^2(1 - 4\sin^2\theta) + \frac{\rho U_1 K_3}{\pi a}\sin\theta.$$

The lift force is

$$-\frac{\rho U_1 K_3}{\pi a}\int_0^{2\pi} a\sin^2\theta\, d\theta = -\rho U_1 K_3.$$

It is instructive to see how the existence of the transverse force may also be established by evaluating excess pressure and momentum flux over a control surface which is far removed from the spindle. Consider, for example, a control surface in the form of a cylinder, coaxial with the spindle but of greater radius b. On this surface the excess pressure is

$$p^*_{r=b} = \frac{1}{2}\rho(U_1^2 - u_r^2 - u_\theta^2),$$

in which the relevant term is

$$\frac{\rho U_1 K_3}{2\pi b}\left(1 + \frac{b^2}{r^2}\right)\sin\theta.$$

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The excess pressure therefore exerts a lift force on the fluid inside the control surface of magnitude

$$-\frac{1}{2}\rho U_1 K_3\left(1 + \frac{a^2}{b^2}\right) \qquad (4.58)$$

per unit length (which reduces to $-\rho U_1 K_3$, of course, when $b = a$). However, the rate of change of momentum of the contained fluid, in the direction of the lift force and per unit length, is the value at $r = b$ of

$$\int_0^{2\pi} \rho u_r(u_r\cos\theta + u_\theta\sin\theta) r\, d\theta$$

[(2.44)], and when this integral is evaluated using (4.57) it is found to amount to

$$\frac{1}{2}\rho U_1 K_3\left(1 - \frac{a^2}{b^2}\right). \qquad (4.59)$$

The difference between (4.58) and (4.59) reveals that the fluid must be subject to an additional force of magnitude $\rho U_1 K_3$ per unit length, which only the spindle can exert upon it. Hence the spindle must experience the reaction which (4.55) describes.

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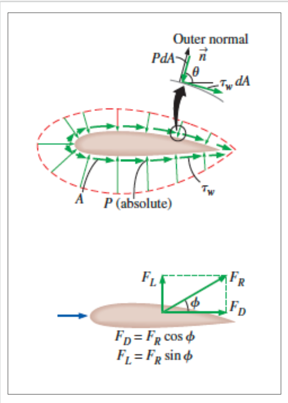

The pressure and viscous forces acting on a two-dimensional body and the resultant lift, F_L and drag, F_D forces.

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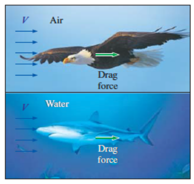
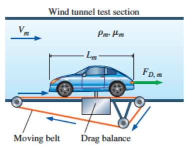
Drag force

- In a real flow, the pressure on the back surface of the body is significantly less than that on the front surface, leading to a nonzero pressure drag on the body. In addition, the no-slip condition on the body surface leads to a nonzero viscous drag as well.
- Thus, the irrotational flow falls short in its prediction of aerodynamic drag for two reasons: it predicts no pressure drag and it predicts no viscous drag.





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